Mathematical Foundations of Infinite-Dimensional Statistical Models

Chap. 4.2 Orthonormal Wavelet Bases

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Contents

Overview

- We present some of the basic ingredients of wavelet theory.
- We focus on the one-dimensional theory.

Recall

- Inner product: $< f, g >= \int_A f(x)\overline{g(x)}dx$ for a set $A \subset \mathbb{R}$.
- Fourier transform: For a function $f \in L^1(\mathbb{R})$,

$$\mathcal{F}[f](u) \equiv \hat{f}(u) = \int_{\mathbb{R}} f(x)e^{-iux}dx, \quad u \in \mathbb{R}.$$

- Orthogonal complement of $V \subseteq H$:

  $V^{\perp} = \{w \in H :< w, v >= 0, \forall v \in V\}$ and $H = V \oplus V^{\perp}$, the (inner) direct sum.

4.2.1 Multiresolution Analysis of L^2

- **Definition 4.2.1** We say that $\phi \in L^2(\mathbb{R})$ is the scaling function of a multiresolution analysis (MRA) of $L^2(\mathbb{R})$ if
  (a) The family $\{\phi(\cdot - k) : k \in \mathbb{Z}\}$ is an ortho-normal system in $L^2(\mathbb{R})$; that is, $< \phi(\cdot - k), \phi(\cdot - l) >= \begin{cases} 1 & \text{when } k = l, \\ 0 & \text{o.w.} \end{cases}$
  (b) The linear spaces

$$V_0 = \left\{ f = \sum_{k\in\mathbb{Z}} c_k \phi(\cdot - k), \{c_k\}_{k\in\mathbb{Z}} : \sum_{k\in\mathbb{Z}} c_k^2 < \infty \right\}, \ldots,$$
$$V_j = \left\{ h = f\left(2^j(\cdot)\right) : f \in V_0 \right\}, \ldots$$

  are nested; that is, $V_{j-1} \subset V_j$ for every $j \in \mathbb{N}$.
  (c) The union $\bigcup_{j\geq 0} V_j$ is dense in L^2.

4.2.1 Multiresolution Analysis of L^2

- Examples for ϕ generating a MRA

  Haar $\phi = 1_{(0,1]}$, V_j equals the space of fts that are piecewise const. on $(k/2^j, (k+1)/2^j]$

  Shannon $\phi(x) = \sin(\pi x)/(\pi x)$ and $V_j = \mathcal{V}_{2^j\pi}$ ($\mathcal{V}_\pi$ is the space of conti. fts $f \in L^2$ which have $\hat{f}$ supported in $[-\pi, \pi]$.)

- ϕ generating the Haar basis is localised in time but not in frequency, ϕ generating the Shannon basis is localised in frequency but not in time.

4.2.1 Multiresolution Analysis of L^2

- Good localisation properties of ϕ could be achieved in time and frequency simultaneously, in the flavour of a Littlewood-Paley decomposition, but without loosing the ortho-normal basis property.

Simple properties of a MRA of L^2

- Since V_j are nested, we can define W_j as the orthogonal complement of V_j in V_{j+1}:

$$W_j = V_{j+1} \ominus V_j \quad \text{or} \quad V_{j+1} = W_j \oplus V_j$$

Then V_j can be written as

$$V_j = V_0 \oplus \left(\bigoplus_{l=0}^{j-1} W_l \right).$$

- For $f \in L^2$, we want to find the orthogonal L^2-projection onto V_j.
    1. The projection of f onto V_0: $K_0(f)(x) = \sum_{k\in\mathbb{Z}} \langle \phi_k, f \rangle \, \phi_k(x)$,
       where $\phi_k = \phi(\cdot - k)$.
    2. To describe the projection onto W_l, assume that there exists a fixed $\psi \in L^2(\mathbb{R})$ s.t., for every $l \in \mathbb{N} \cup \{0\}$,

$$\left\{ \psi_{lk} := 2^{l/2} \psi \left(2^l(\cdot) - k \right) : k \in \mathbb{Z} \right\}$$

       is an ortho-normal set of functions that spans W_l.
       $\rightarrow$ The projection of f onto W_l: $\sum_k \langle \psi_{lk}, f \rangle \, \psi_{lk}$
    3. The projection $K_j(f)$ of f onto V_j:

$$K_j(f)(x) = \sum_{k\in\mathbb{Z}} \langle \phi_k, f \rangle \, \phi_k(x) + \sum_{l=0}^{j-1} \sum_{k\in\mathbb{Z}} \langle \psi_{lk}, f \rangle \, \psi_{lk}(x)$$

- If $\bigcup_{j\geq 0} V_j$ is dense in L^2, the space L^2 can be decomposed into the direct sum,

$$L^2 = V_0 \oplus \left(\bigoplus_{l=0}^{\infty} W_l \right)$$

so the set of fts

$$\left\{ \phi(\cdot - k), 2^{l/2}\psi\left(2^l(\cdot) - k\right) : k \in \mathbb{Z}, l \in \mathbb{N} \cup \{0\} \right\}$$

is an ortho-normal **wavelet** basis of the Hilbert space L^2.

- Every $f \in L^2$ has the **wavelet series** expansion:

$$f = \sum_{k \in \mathbb{Z}} \langle \phi_k, f \rangle \, \phi_k + \sum_{l=0}^{\infty} \sum_{k \in \mathbb{Z}} \langle \psi_{lk}, f \rangle \, \psi_{lk}$$

where convergence is guaranteed at least in the L^2.

- Theorem 4.2.2 gives sufficient conditions to construct ϕ, ψ in the Fourier domain.
- **Theorem 4.2.2** Let $\phi \in L^2(\mathbb{R}), \phi \neq 0$.
  (a) $\{\phi(\cdot - k) : k \in \mathbb{Z}\}$ forms an ortho-normal system in L^2
  $\Leftrightarrow \sum_{k\in\mathbb{Z}} |\hat{\phi}(u + 2\pi k)|^2 = 1$ a.e.
  (b) Suppose that $\{\phi(\cdot - k) : k \in \mathbb{Z}\}$ forms an ortho-normal system in L^2.
  Then corresponding $(V_j)_{j\in\mathbb{Z}}$ are nested
  $\Leftrightarrow$ there exists a 2π-periodic ft $m_0 \in L^2((0, 2\pi])$ s.t.

$$\hat{\phi}(u) = m_0\left(\frac{u}{2}\right)\hat{\phi}\left(\frac{u}{2}\right) \quad \text{a.e.} \tag{4.34}$$

- **Theorem 4.2.2** (cont'd)

  (c) Let ϕ be a scaling ft satisfying (a) and (b) of Def. 4.2.1, and let m_0 satisfy (4.34). If $\psi \in L^2$ satisfies

$$\hat{\psi}(u) = m_1\left(\frac{u}{2}\right)\hat{\phi}\left(\frac{u}{2}\right) \quad \text{a.e.} \tag{4.35}$$

  where $m_1(u) = \overline{m_0(u+\pi)}e^{-iu}$, then ψ is a wavelet function;

  that is, $\{\psi(\cdot - k) : k \in \mathbb{Z}\}$ forms an ortho-normal basis of $W_0 = V_1 \ominus V_0$, and any $f \in V_1$ can be uniquely decomposed as $\sum_k c_k\phi(\cdot - k) + \sum_k c_k'\psi(\cdot - k)$ for sequences $\{c_k\}, \{c_k'\} \in \ell_2$.

Properties of ϕ, ψ

- **Corollary 4.2.4** For ϕ a scaling ft generating a MRA of L^2, we have

$$\phi(x) = \sqrt{2}\sum_k h_k \phi(2x-k) \text{ a.e.}, \quad h_k = \sqrt{2}\int_{\mathbb{R}} \phi(x)\overline{\phi(2x-k)}dx \tag{4.39}$$

and

$$\psi(x) = \sqrt{2}\sum_k \lambda_k \phi(2x-k) \text{ a.e.}, \quad \lambda_k = (-1)^{k+1}\overline{h}_{1-k} \tag{4.40}$$

Moreover, if $\int_{\mathbb{R}} \phi(x)dx = 1$, then

$$\sum_k \overline{h}_k h_{k+2l} = \delta_{0l}, \quad \frac{1}{\sqrt{2}}\sum_k h_k = 1 \tag{4.41}$$

4.2.2 Approximation with Periodic Kernels

- The preceding results gives conditions to verify (a), (b) from Def. 4.2.1.
- But, they did not verify whether the $\{V_j\}_{j \geq 0}$ are dense in L^2.
- This can be verified by showing the projection kernel satisfies Condition 4.1.4, so that then $K_{2^{-j}}(f) \to f$ in L^2 as $j \to \infty$;
  - Projection kernel: $K(x, y) = \sum_k \phi(x - k)\phi(y - k)$
  - $K_h(f) = \frac{1}{h} \int_{\mathbb{R}} K\left(\frac{\cdot}{h}, \frac{y}{h}\right) f(y)dy, \ h > 0.$

4.2.2 Approximation with Periodic Kernels

- **Condition 4.1.4** Let K be a m'ble ft $K(x,y) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. For $N \in \mathbb{N}$, assume that

  (M) $c_N(K) \equiv \int_{\mathbb{R}} \sup_{v \in \mathbb{R}} |K(v, v-u)||u|^N du < \infty$

  (P) For every $v \in \mathbb{R}$ and $k = 1, \ldots, N-1$,

$$\int_{\mathbb{R}} K(v, v+u) du = 1 \quad \text{and} \quad \int_{\mathbb{R}} K(v, v+u) u^k du = 0.$$

- **Proposition 4.2.6** Assume that
  - for some nonincreasing ft $\Phi \in L^\infty([0,\infty)) \cap L^1([0,\infty))$, we have $\phi(u) \leq \Phi(|u|), \forall u \in \mathbb{R}$ and $\int_{\mathbb{R}} \Phi(|u|)|u|^N du < \infty$ for some $N \in \mathbb{N} \cup \{0\}$;
  - as $u \to 0$, $|\hat{\phi}(u)|^2 = 1 + o\left(|u|^N\right), \hat{\phi}(u + 2\pi k) = o\left(|u|^N\right) \quad \forall k \neq 0.$

  Then $\left|\int_{\mathbb{R}} \phi(x) dx\right| = 1$ and for every $l = 1, \ldots, N$ and almost every $x \in \mathbb{R}$, we have

$$\int_{\mathbb{R}} K(x, x+u) du = 1 \quad \text{and} \quad \int_{\mathbb{R}} K(x, x+u) u^l du = 0 \tag{4.46}$$

- For Condition 4.1.4 (M), use the fact that, for Φ in Prop 4.2.6, $\sup_{v \in \mathbb{R}} |K(v, v-u)| \leq c_1 \Phi\left(c_2 |u|\right)$ for some $0 < c_1, c_2 < \infty$ and every $u \in \mathbb{R}$ (which is in Prop. 4.2.5).

- Combining Propositions 4.1.3, 4.2.5, 4.2.6, we conclude that

$$\|K_{2^{-j}}(f) - f\|_p \to 0$$

whenever $f \in L^p$, $1 \leq p < \infty$.

- If $K_{2^{-j}}$ is the projector onto V_j then $\{V_j\}_{j\geq 0}$ are dense in L^p.
- **Proposition 4.2.7** Let ϕ, ψ be as in Thm. 4.2.2, part (c), and suppose that ϕ satisfies the conditions of Prop. 4.2.6 for some N. Then

$$\int_{\mathbb{R}} \psi(x) x^l dx = 0 \quad \forall l = 0, \ldots, N$$

- **Proposition 4.2.8** Suppose that $\phi \in L^1(\mathbb{R})$ is s.t. $\sup_{x\in\mathbb{R}} \sum_{k\in\mathbb{Z}} |\phi(x-k)| \equiv \kappa < \infty$. Let $c \equiv \{c_k : k \in \mathbb{Z}\} \in \ell_p$, $1 \le p \le \infty$. Then,
  - for every $l \ge 0$ and some const. $K = K(\kappa, \|\phi\|_1, p)$, we have

$$\left\| \sum_{k\in\mathbb{Z}} c_k 2^{l/2} \phi\left(2^l \cdot - k\right) \right\|_p \le K \|c\|_p 2^{l(1/2-1/p)}$$

  - if, moreover, the set $\{\phi(\cdot - k) : k \in \mathbb{Z}\}$ is ortho-normal in L^2, then for some const. $K' = K'(\kappa, \|\phi\|_1, p)$,

$$\left\| \sum_{k\in\mathbb{Z}} c_k 2^{l(1/2-1/p)} \phi\left(2^l \cdot - k\right) \right\|_p \ge K' \|c\|_p 2^{\iota(1/2-1/p)}.$$

4.2.3 Construction of Scaling Functions

- Construct scaling fts ϕ that generate a MRA of $L^2(\mathbb{R})$ as in Def. 4.2.1 and whose projection kernels have good approximation properties.
- Focus on tow main examples of scaling fts and wavelets:
  - $\hat{\phi}, \hat{\psi}$ have compact support $\rightarrow$ "Band-Limited Wavelets"
  - ϕ, ψ have compact support $\rightarrow$ "Daubechies Wavelets"

Band-Limited Wavelets

- We construct band-limited wavelets as follows:
  - Take μ any prob. measure supported in a closed subinterval of $[-\pi/3, \pi/3]$;
  - Define ϕ by

$$\hat{\phi}(u) = \sqrt{\int_{u-\pi}^{u+\pi} d\mu}, \tag{4.51}$$

  for $u \in [-4\pi/3, 4\pi/3]$. Then, $\hat{\phi}(u) = 1$ for $u \in (-2\pi/3, 2\pi/3)$, and $\int \phi = \hat{\phi}(0) = 1$.

1 Check the first condition of Def 4.2.1 in view of Thm. 4.2.2:

$$\sum_{k \in \mathbb{Z}} |\hat{\phi}(u + 2\pi k)|^2 = \sum_{k \in \mathbb{Z}} \int_{u+(2k-1)\pi}^{u+(2k+1)\pi} d\mu = \int_{\mathbb{R}} d\mu = 1$$

Band-Limited Wavelets

2 For second condition, we set for $u \in [-2\pi, 2\pi]$,

$$m_0(u/2) = \begin{cases} \hat{\phi}(u) & |u| \leq 4\pi/3 \\ 0 & 4\pi/3 < |u| \leq 2\pi \end{cases}$$

Then we have $\hat{\phi}(u) = m_0\left(\frac{u}{2}\right)\hat{\phi}\left(\frac{u}{2}\right)$ a.e..

3 Third condition is checked by Prop 4.2.6 and using follows:

- Since $\hat{\phi}$ is identically 1 near the origin, we have $|\hat{\phi}(u)|^2 = 1 + o\left(|u|^N\right)$ for every N;
- Since $\hat{\phi}$ is supported in $[-4\pi/3, 4\pi/3]$, we have for $|u|$ small enough and every N, $\hat{\phi}(u + 2\pi k) = 0 = o\left(|u|^N\right)$ whenever $k \neq 0$.

- **Theorem 4.2.9** There exists a band-limited ortho-normal multiresolution wavelet basis

$$\{\phi_k = \phi(\cdot - k), \psi_{lk} = 2^{l/2}\psi(2^l(\cdot) - k) : k \in \mathbb{Z}, l \in \mathbb{N} \cup \{0\}\}$$

of $L^2(\mathbb{R})$ with scaling function $\phi \in \mathcal{S}(\mathbb{R}), \int \phi = 1$ wavelet $\psi \in \mathcal{S}(\mathbb{R}), \int \psi = 0$ s.t.

(a) $supp(\hat{\phi}) \subset \{u : |u| \leq 4\pi/3\}, \quad supp(\hat{\psi}) \subset \{u : |u| \in [2\pi/3, 4\pi/3]\}$,

(b) $\int_{\mathbb{R}} \psi(u) u^l du = 0, \forall l \in \mathbb{N} \cup \{0\}$ and for all $v \in \mathbb{R}, l \in \mathbb{N}$,
$\int_{\mathbb{R}} K(v, v+u) du = 1 \quad$ and $\quad \int_{\mathbb{R}} K(v, v+u) u^l du = 0$,

(c) $\sum_{k\in\mathbb{Z}} |\phi(\cdot - k)| \in L^\infty(\mathbb{R}), \quad \sum_{k\in\mathbb{Z}} |\psi(\cdot - k)| \in L^\infty(\mathbb{R})$ and

(d) For $\kappa(x, y)$ equal to either $K(x, y)$ or $\sum_k \psi(x-k)\psi(y-k)$,
$\sup_{v\in\mathbb{R}} |\kappa(v, v-u)| \leq c_1 \Phi(c_2|u|)$ for some $0 < c_1, c_2 < \infty, \forall u \in \mathbb{R}$,
for some bdd ft $\Phi : [0, \infty) \to \mathbb{R}$ that decays faster than any inverse polynomial at $+\infty$.

Daubechies Wavelets

- **Theorem 4.2.10** For every $N \in \mathbb{N}$, there exist an ortho-normal multiresolution wavelet basis
  $\{\phi_k = \phi(\cdot - k), \psi_{lk} = 2^{l/2}\psi(2^l(\cdot) - k) : k \in \mathbb{Z}, l \in \mathbb{N} \cup \{0\}\}$ of $L^2(\mathbb{R})$
  with scaling function $\phi \equiv \phi^{(N)}, \int \phi = 1$ wavelet $\psi \equiv \psi^{(N)}, \int \psi = 0$
  s.t.

  (a) $supp(\phi) \subset \{x : 0 \leq x \leq 2N - 1\}, \quad supp(\psi) \subset \{x : -N + 1 \leq x \leq N\}$,

  (b) $\int_{\mathbb{R}} \psi(u) u^l du = 0, \forall l = 0, 1, \ldots, N - 1$ and for all
  $v \in \mathbb{R}, l = 1, \ldots, N - 1$,
  $\int_{\mathbb{R}} K(v, v + u) du = 1$ and $\int_{\mathbb{R}} K(v, v + u) u^l du = 0$,

  (c) $\sum_{k \in \mathbb{Z}} |\phi(\cdot - k)| \in L^\infty(\mathbb{R}), \quad \sum_{k \in \mathbb{Z}} |\psi(\cdot - k)| \in L^\infty(\mathbb{R})$,

  (d) For $\kappa(x, y)$ equal to either $K(x, y)$ or $\sum_k \psi(x - k)\psi(y - k)$,
  $\sup_{v \in \mathbb{R}} |\kappa(v, v - u)| \leq c_1 \Phi(c_2 |u|)$ for some $0 < c_1, c_2 < \infty, \forall u \in \mathbb{R}$,
  for some bdd and compactly supported ft $\Phi : [0, \infty) \to \mathbb{R}$, and

  (e) For $N \geq 2$, ϕ, ψ are elements of $C^{[\lambda(N-1)]}(\mathbb{R})$ for some $\lambda \geq 0.18$.